How Are Futures Returns Already Excess of ZCB Returns?

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We will show that each day's futures PNL is excess of the ZCB return, using the simplifying assumption that there is no futures convexity adjustment so that we can operate with the forward price. First an aside that I don't really get why futures contracts are settled every day with the full difference between today's futures price F_t and yesterday's futures price F_{t-1} exchanging hands. If it were a forward contract, then the fair value of the contract today would be not $F_t - F_{t-1}$ but rather $(F_t - F_{t-1})e^{-r_t \tau}$, so in my opinion that's the quantity that should exchange hands. Notice that the "proper" cashflow is discounted back to today, which makes sense: The forward price represents a cashflow at time T, not time t. But OK. Probably this is part of what leads to the requirement that we use the convexity adjustment if we want to be totally correct.

Definitions

- T some future time, which we hold fixed so that our notation will be simpler
- $\tau := T - t$
- r_t the (logarithmic) interest rate at time t over the period from t to T (this notation cannot handle forward rates, but that's OK for us)
- $Z_t := e^{-r_t \tau}$ the price at time t of the zero-coupon bond (ZCB) that pays out $1 at time T
- $z_t := \ln\left(\frac{Z_{t+1}}{Z_t}\right)$ the (logarithmic) return of the ZCB between time t and time $t+1$, so that if you invested $\$N$ in ZCB's at time t then you'd have $\$Ne^{z_t}$ at time $t+1$
- S_t the underlying spot price at time t
- $s_t := \ln(\left(\frac{S_{t+1}}{S_t}\right)$ the (logarithmic) underlying spot return between time t and time T
- $F_t := S_t e^{r_t \tau} = \frac{S_t}{Z_t}$ the forward price at time t for a contract that matures at time T

Arithmetic

We have that the (arithmetic, i.e. in dollars) futures PNL (assume we are long) tomorrow will be

$$\begin{aligned}
F_{t+1} - F_t &= S_{t+1} e^{r_{t+1}(\tau-1)} - S_t e^{r_t \tau} \\
&= S_t \left(\frac{S_{t+1}}{S_t} e^{r_{t+1}(\tau-1)} - e^{r_t \tau} \right) \\
&= S_t e^{r_t \tau} \left(\frac{S_{t+1}}{S_t} e^{r_{t+1}(\tau-1) - r_t \tau} - 1 \right) \\
&= F_t \left(\frac{S_{t+1}}{S_t} \frac{e^{r_{t+1}(\tau-1)}}{e^{r_t \tau}} - 1 \right) \\
&= F_t \left(\frac{S_{t+1}}{S_t} \frac{Z_t}{Z_{t+1}} - 1 \right) \\
&= F_t \left(e^{s_t} e^{-z_t} - 1 \right) \\
&= F_t \left(e^{s_t - z_t} - 1 \right),
\end{aligned}$$

so if we define our "principal" (AKA capital base AKA notional exposure) to a futures contract as the futures price F_t then our geometric return is

$$\frac{F_t(e^{s_t - z_t} - 1)}{F_t} = e^{s_t - z_t} - 1,$$

which in logarithmic terms is a return of

$$\ln(1 + e^{s_t - z_t} - 1) = s_t - z_t,$$

QED.

Recovering the Total Return

Of course, a futures agreement is free to enter, so defining our principal as F_t suggests that we simply have that much cash somewhere, perhaps stuffed under a mattress. If we instead invested that cash in ZCB's at time t ("investing full margin" AKA "hedging out interest rate risk"), then our portfolio at time $t+1$ would be worth

$$= F_t\left(e^{s_t - z_t} - 1\right) + F_t e^{z_t},$$

with the first part coming from the futures PNL and the second part coming from the ZCB, so that our geometric return would be

$$\begin{aligned} &= \frac{F_t\left(e^{s_t - z_t} - 1\right) + F_t e^{z_t}}{F_t} - 1 \\ &= e^{s_t - z_t} - 1 + e^{z_t} - 1 \\ &= e^{s_t - z_t} - 1 + e^{z_t} - 1, \end{aligned}$$

whence using the approximation $e^x \approx 1 + x$ for x small,

$$\begin{aligned} &\approx 1 + (s_t - z_t) - 1 + 1 + z_t - 1 \\ &= (s_t - z_t) + z_t \\ &= s_t, \end{aligned}$$

as desired. Of course, this is unsatisfying for two applications of the approximation that logarithmic return is the same as geometric return for small returns: (1) the one we explicitly used above, and (2) the one where we were happy based on that first application of the approximation that our geometric return ends up being s_t, which is actually the desired *logarithmic* not geometric return.

In order to truly recover s_t the total return of the underlying spot price, we would need to know s_t in advance. But the actual expression for how much to invest in ZCB's in that case is ugly and I haven't yet simplified it into any expression worth presenting (and by the way I'm not even sure if it can be simplified into such a form).